NAUB JOURNAL OF SCIENCE AND TECHNOLOGY
(NAUBJOST)
e-ISSN:2811-2350
Email: jost@naub.edu.ng http://journal.naub.edu.ng/naubjournal/

SOFT BOOLEAN ALGEBRA

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Abstract

This paper crisply presents the fundamentals of soft set theory to emphasize that soft set has enough developed basic supporting tools through which various algebraic structures in theoretical point of view could be developed. We redefined the concepts of conjunction and disjunction as binary operations on soft sets and present their properties. In particular, a perception named soft Boolean algebra is introduced where some related results were established.

Keywords: conjunction, disjunction, Parameters, soft Boolean algebra, Soft set, and universe set.

Introduction

Most of the challenges we encountered in real life such as in engineering, economics, environmental sciences, medical and social sciences have various degrees of uncertainties and imprecision embedded in them. Solutions to these problems lie in the use of mathematical principles based on uncertainty and imprecision. In an attempt to proffer solutions to these problems different theories were formulated, such as theory of probability (Prade and Duboise, 1980), theory of fuzzy set (Zadeh, 1965), theory of interval mathematics (Atanassov, 1994), theory of rough sets (Pawlak, 1982) and theory of vague sets (Gau and Buehrer, 1993), which were considered as mathematical tools for handling uncertainties and imprecision. But all these theories have their short comings in dealing with uncertainty. One major setback associated with these theories is the inadequacies of the parameterization tools. To overcome these limitations (Molodtsov, 1999) introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties and imprecision that is free from the difficulties suffered by the existing classical mathematical approaches. Thereafter some new operations on soft set were introduced by (Irfan *et al.*, 2009), these operations have really enhanced the study of soft set.

Research on soft sets has been progressing rapidly, since its introduction by Molodtsov in 1999 up to the present time and several results have been achieved both in theory and applications. As shown by the literature developed by (Ibrahim and Yusuf, 2012).

Work on algebraic structures of soft sets were carried out by (Aktas and Cagman, 2007) on soft set and soft group, (Atagun and Sezgin, 2011) introduced soft substructures of rings, field and modules, (Acar *et al.*, 2010) worked on soft set and soft ring and established some results. Similarly, (Feng, *et al.*, 2009) introduced the study of soft semiring and presented some basic results.

In this research paper, we redefined the concept of conjunction and disjunction as binary operations on soft sets and explicitly present their various algebraic properties. The redefined concept of conjunction and disjunction operations enable us to introduce the notion of soft Boolean Algebra.

Preliminaries

Let U be a universal set and E be the set of all possible parameters under consideration with respect to U. Let the power set of U (i.e., the set of all subsets of U) be denoted by $P(U)$ and A is a subset of the parameters, E ($A \subseteq E$). The parameters are attributes, characteristics or properties associated with the objects in U.

Definition 2.1

A pair (F, E) is called a soft set over U if and only if F is a mapping of E into the set of all subsets of the set U. That is, a soft set is a parametrized family of subsets of the set U. For all $e \in E$, $F(e)$ is considered as the set of e −approximate elements of the soft set (F, E).

Definition 2.2

A soft set (F, E) over a universe U is said to be *null* or *empty* soft set denoted by $\widetilde{\emptyset}$, if $\forall e \in E, F(e) = \emptyset$.

Definition 2.3

A soft set (F, A) over a universe U is called *absolute* or *universal soft set* denoted by $\widetilde{(F, A)}$ or $\widetilde{U}$, if $\forall e \in E, F(e) = U$.

Definition 2.4

Let $E = \{e_1, e_2, e_3, \ . \ . \ ., e_n\}$ be a set of parameters. The *not-set* of E denoted by $\neg E$ is defined as $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \ . \ . \ ., \neg e_n)$.

Definition 2.5

The *complement* of a soft set (F, E), denoted by $(F, E)^c$, is defined as $(F, E)^c = (F^c, \neg E)$.

Where $F^c: \neg E \rightarrow P(U)$ is a mapping given by $F^c(\alpha) = U - F(\neg\alpha), \forall\alpha\in\neg E$. F^c is called the soft complement function of F. Consequently, $(F^c)^c = F$ and $((F, E)^c)^c = (F, E)$

Definition 2.6

Let (F, A) and (G, B) be any two soft sets over a common universe U, (F, A) is called a soft subset of (G, B), denoted by $(F, A) \widetilde{\subseteq} (G, B)$ if ;

(i) $A \subset B$, and
(ii) $\forall e \in A, F(e) = G(e)$

(F, A) is said to be a soft super set of (G, B), if (G, B) is a subset of (F, A) and it is denoted by $(F, A) \widetilde{\supseteq} (G, B)$.

Definition 2.7

Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal, denoted by $(F, A) = (G, B)$, if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.8
If (F,A) and (G,B) are two soft sets then "(F,A) AND (G,B)" denoted by $(F,A) \wedge (G,B)$ is defined as $(F,A) \wedge (G,B) = (H,\ A \times B)$, where $H(\alpha,\beta) = F(\alpha) \cap G(\beta)$, $\forall(\alpha,\beta) \in A \times B$.

Definition 2.9
If (F,A) and (G,B) are two soft sets then "$(F,\ A)$ OR (G,B)" denoted by $(F,A) \vee (G,B)$ is defined as $(F,\ A) \vee (G,B) = (P,\ A \times B)$, where, $P(\alpha,\beta) = F(\alpha) \cup G(\beta), \forall\ (\alpha,\beta) \in A \times B$.

Definition 2.10
Let (F,A) and (G,B) be two soft sets over a common universe U. The union or extended union of (F,A) and (G,B), denoted by $(F,A) \cup (G,B)$ or $(F,A) \cup_E (G,B)$, is the soft set (H,C) satisfying the following conditions:
(i) $C = A \cup B$, (ii) $\forall\ e \in C,\ \ H(e) = \begin{cases} F(e) & \text{if } e \in A\backslash B \\ G(e) & \text{if } e \in B\backslash A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$

Definition 2.11
The intersection of two soft sets (F,A) and (G,B) over a common universe set U is the soft set (H,C), where $C = A \cap B$, and $\forall e \in C, H(e) = F(e)$ or $G(e)$, we write $(F,A) \cap (G,B) = (H,C)$

Definition 2.12
The extended intersection of soft sets $(F,A)$and (G,B) over a common universe U, denoted by $(F,A) \cap_E (G,B)$, is the soft set (H,C), where $C = A \cup B$ $\forall e \in C$ and

$$H(e) = \begin{cases} F(e) & \text{if } e \in A\backslash B \\ G(e) & \text{if } e \in B\backslash A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}$$

Definition 2.13
The restricted intersection of soft sets $(F,A)$and (G,B) over a common universe U, denoted by $(F,A) \cap_R (G,B)$, is the soft set (H,C), where $C = A \cap B \neq \emptyset$ such that $H(e) = F(e) \cap G(e),\ \forall e \in C$.

Definition 2.14

Let (F,A) and (G,B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The restricted union of (F,A) and (G,B), denoted by $(F,A)\ \cup_R\ (G,B)$, is defined as $(F,A)\ \cup_R\ (G,B) = (H,C)$, where $C = A \cup B$, and $\forall e \in C, H(e) = F(e) \cup G(e)$.

Definition 2.15

Let (F,A) and (G,B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$ .The restricted difference of (F,A) and (G,B) denoted by $(F,A)\ \sim_R\ (G,B)$, is defined as $(F,A)\ \sim_R\ (G,B) = (H,C)$, where $C = A \cap B$, and $\forall e \in C, H(e) = F(e) \setminus G(e)$.

Definition 2.16

The restricted symmetric difference of two soft sets (F,A) and (G,B) over a common universe U is defined as $(F,A)\,\Delta(G,B) = (F,A)\ \cup_R\ (G,B) \sim_R \big((G,B)\ \cup_R\ (F,A)\big)$

Redefined Concept of Conjunction (AND) and Disjunction (OR)

Definition 2.8 and 2*.9* are redefined as follows:

Definition 3.1
Given any two soft sets (F,A) and (G,B) over a common universe U, the disjunction of (F,A) and (G,B) denoted by $(F,A) \wedge (G,B)$ is defined as $(F,A) \wedge (G,B) = (H,\ A \cap B)$, where $H(\alpha) = F(\alpha) \cap G(\beta), \forall \alpha \in A \cap B$.

Definition 3.2
Given any two soft sets (F,A) and (G,B) over a common universe U, the conjunction of (F,A) and (G,B) denoted by $(F,A) \vee (G,B)$ is defined as $(F,\ A) \vee (G,B) = (P,\ A \cup B)$, where $P(\alpha) = F(\alpha) \cup G(\beta), \forall\ \alpha \in A \cup B$.

Example
Let $U = \{h_1, h_2, h_3, \dots, h_8\}$ be a universe. Let $A = \{e_1, e_2, e_3\}$, $B = \{e_1, e_3, e_4\}$, $C = \{e_2, e_3, e_5\}$ be any given sets of parameters, such that

$$(F,A) = \{F(e_1) = \{h_1, h_2, h_3\},\ F(e_2) = \{h_2, h_3\},\ F(e_3) = \{h_4\}\}$$
$$(G,B) = \{G(e_1) = \{h_1, h_3, h_4\}, G(e_3) = \{h_3, h_4\}, G(e_4) = \{h_4\}\}$$
$$(H,C) = \{H(e_2) = \{h_1, h_4, h_5\},\ H(e_3) = \{h_3\},\ H(e_5) = \{h_1, h_2\}\}$$

$(F,A) \vee ((G,B) \vee (H,C)) = (F,A) \vee (T, B \cup C)$, where $(B \cup C) = \{e_1, e_2, e_3, e_4, e_5\}$,

$$(T, B \cup C) = \begin{Bmatrix} T(e_1) = \{h_1, h_3, h_4\}, T(e_2) = \{h_1, h_4, h_5\}, T(e_3) = \{h_3, h_4\}, T(e_4) = \{h_4\}, \\ T(e_5) = \{h_1, h_2\} \end{Bmatrix}$$

$(F,A) \vee (T, B \cup C) = (N, A \cup (B \cup C))$, where $A \cup (B \cup C) = \{e_1, e_2, e_3, e_4, e_5\}$,

$$(F,A) \vee (T, B \cup C) = \begin{Bmatrix} L(e_1) = \{h_1, h_2, h_3, h_4\}, L(e_2) = \{h_1, h_2, h_3, h_4, h_5\}, L(e_3) = \{h_3, h_4\}, \\ L(e_4) = \{h_4\}, L(e_5) = \{h_1, h_2\} \end{Bmatrix}$$

$$= (F,A) \vee ((G,B) \vee (H,C)).$$

Also, for $((F,A) \vee (G,B)) \vee (H,C)$,

$(F,A) \vee (G,B) = (P, A \cup B)$, where $(A \cup B) = \{e_1, e_2, e_3, e_4\}$

$$(F,A) \vee (G,B) = (P, A \cup B) = \begin{Bmatrix} P(e_1) = \{h_1, h_2, h_3, h_4\}, P(e_2) = \{h_2, h_3\}, P(e_3) = \{h_3, h_4\}, \\ P(e_4) = \{h_4\} \end{Bmatrix}$$

$(P, A \cup B) \vee (H,C) = (L, (A \cup B) \cup C)$, where $(A \cup B) \cup C = \{e_1, e_2, e_3, e_4, e_5\}$

$$(P, A \cup B) \vee (H,C) = \begin{Bmatrix} L(e_1) = \{h_1, h_2, h_3, h_4\}, L(e_2) = \{h_1, h_2, h_3, h_4, h_5\}, L(e_3) = \{h_3, h_4\}, \\ L(e_4) = \{h_4\}, L(e_5) = \{h_1, h_2\} \end{Bmatrix}$$

$$= ((F,A) \vee (G,B)) \vee (H,C)$$

Consequently, $(F,A) \vee ((G,B) \vee (H,C)) = ((F,A) \vee (G,B)) \vee (H,C)$.

Proposition 3.1 **Properties of the Redefined Conjunction ($\wedge$) and Disjunction ($\vee$)**
Given any soft sets (F, A), (G, B) and (H, C) over a common universe *U,* then the following properties are satisfied:

a. Idempotent laws

(i) $(F, A) \vee (F, A) = (F, A)$

(ii) $(F, A) \wedge (F, A) = (F, A)$

b. Commutativity laws

(i) $(F, A) \vee (G, B) = (G, B) \vee (F, A)$

(ii) $(F, A) \wedge (G, B) = (G, B) \wedge (F, A)$

c. Associativity laws

(i) $\big((F, A) \vee (G, B)\big) \vee (H, C) = (F, A) \vee \big((G, B) \vee (H, C)\big)$

(ii) $\big((F, A) \wedge (G, B)\big) \wedge (H, C) = (F, A) \wedge \big((G, B) \wedge (H, C)\big)$

d. Distributive laws

(i) $(F, A) \vee \big((G, B) \wedge (H, C)\big) = \big((F, A) \vee (G, B)\big) \wedge \big((F, A) \vee (H, C)\big)$

(ii) $(F, A) \wedge \big((G, B) \vee (H, C)\big) = \big((F, A) \wedge (G, B)\big) \vee \big((F, A) \wedge (H, C)\big)$

e. Absorption laws

(i) $(F, A) \vee \big((F, A) \wedge (G, B)\big) = (F, A)$

(ii) $(F, A) \wedge \big((F, A) \vee (G, B)\big) = (F, A)$

f. Complement laws

(i) $(F, A) \vee (F, A)^c = \widetilde{U}$

(ii) $(F, A) \wedge (F, A)^c = \widetilde{\emptyset}$

g. Identity laws

(i) $(F, A) \vee \widetilde{\emptyset} = (F, A)$

(ii) $(F, A) \wedge \widetilde{\emptyset} = \widetilde{\emptyset}$

h. Universe laws

(i) $(F, A) \vee \widetilde{U} = \widetilde{U}$

(ii) $(F, A) \wedge \widetilde{U} = (F, A)$

i. DeMorgan's laws

(i) $\big((F, A) \vee (G, B)\big)^c = (F, A)^c \wedge (G, B)^c$

(ii) $\left((F,\ A) \wedge (G,B)\right)^c = (F,A)^c \vee (G,B)^c$

Proof (c) and (i)

(c). Associativity

Given any soft sets (F,A), (G,B) and (H,C) over a common universe *U,* then (i) $\left((F,A) \vee (G,B)\right) \vee (H,C) = (F,A) \vee \left((G,B) \vee (H,C)\right)$

Proof

LHS

Let $\alpha(x) \in (F,A) \vee \left((G,B) \vee (H,C)\right)$

$\Rightarrow\ x \in A$ or $\alpha(x) \in \left((G,B) \vee (H,C)\right)$

$\Rightarrow\ x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$,

where $\alpha(x) = F(x) \cup (G(x) \cup H(x)), \forall x \in A \cup (B \cup C)$

$= (F(x) \cup G(x)) \cup H(x)$

$\Rightarrow x \in (A \cup B) \cup C$

$\Rightarrow \alpha(x) \in \left((F,A) \vee (G,B)\right) \vee (H,C)$

Hence,

$(F,A) \vee \left((G,B) \vee (H,C)\right) \tilde{\subseteq} \left((F,A) \vee (G,B)\right) \vee (H,C)$ *(1)*

Conversely,

Let $\beta(x) \in \left((F,A) \vee (G,B)\right) \vee (H,C)$

$\Rightarrow \beta(x) \in \left((F,A) \vee (G,B)\right)$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cup C$,

where $\beta(x) = (F(x) \cup G(x)) \cup H(x), \forall x \in (A \cup B) \cup C$

$= F(x) \cup (G(x) \cup H(x))$

$\Rightarrow\ x \in A$ or $\beta(x) \in (G(x) \cup H(x))$

$\Rightarrow\ x \in A$ or $x \in (B \cup C)$

i.e, $x \in A \cup (B \cup C)$,

$\Rightarrow \beta(x) \in (F,A) \vee \left((G,B) \vee (H,C)\right)$

Hence,

$\left((F,A) \vee (G,B)\right) \vee (H,C) \tilde{\subseteq} (F,A) \vee \left((G,B) \vee (H,C)\right)$ *(2)*

From *(1)* and *(2)* we have

$(F,A) \vee \left((G,B) \vee (H,C)\right) = \left((F,A) \vee (G,B)\right) \vee (H,C)$.

Similarly, $(F,A) \wedge \left((G,B) \wedge (H,C)\right) = ((F,A) \wedge (G,B)) \wedge (H,C)$

(i). De Morgan's laws

Given any soft sets (F,A), (G,B) over a common universe *U*, then the following De Morgan laws hold.

(i) $\left((F,A) \vee (G,B)\right)^c = (F,A)^c \wedge (G,B)^c$

From LHS

Let $\alpha(x) \in \left((F,A) \vee (G,B)\right)^c$

$\Rightarrow \alpha(x) \notin (F, A) \vee (G, B)$

$\Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A^c$ and $x \in B^c$,

where $\alpha(x) = F^c(x) \cap G^c(x) = \left(\widetilde{U} - F(x)\right) \cap \left(\widetilde{U} - G(x)\right)$

$\Rightarrow x \in A^c \cap B^c$

$\Rightarrow \alpha(x) \in (F, A)^c \wedge (G, B)^c$.

Hence,

$$\left((F, A) \vee (G, B)\right)^c \cong (F, A)^c \wedge (G, B)^c \qquad (1)$$

Conversely,

From RHS

Let $\beta(x) \in (F, A)^c \wedge (G, B)^c$

$\Rightarrow x \in A^c \cap B^c$,

where $\beta(x) = F^c(x) \cap G^c(x) = \left(\widetilde{U} - F(x)\right) \cap \left(\widetilde{U} - G(x)\right)$

$\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \notin A \cup B$

$\Rightarrow \beta(x) \notin (F, A) \vee (G, B)$

$\Rightarrow \beta(x) \in \left((F, A) \vee (G, B)\right)^c$.

Hence,

$$(F, A)^c \wedge (G, B)^c \cong \left((F, A) \vee (G, B)\right)^c \qquad (2)$$

From *(1)* and *(2)*,

$$\left((F, A) \vee (G, B)\right)^c = (F, A)^c \wedge (G, B)^c$$

Similarly, the proof of the remaining properties of the redefined conjunction and disjunction follow.

Soft Boolean algebra

Theorem 4.1

Let SB be a collection of all soft sets under a common universe U. Let $\emptyset \in SB$ be an empty soft set and $\widetilde{U}$ be the universal soft set. Then SB equipped with the two binary operations $\wedge$ and $\vee$ is a soft Boolean algebra. That is $(SB, \wedge, \vee, \emptyset, \widetilde{U})$ is a soft Boolean algebra.

Proof

Let (F, A), (G, B) and $(H, C) \in (SB, \wedge, \vee, \emptyset, \widetilde{U})$ be soft sets over a common universe *U,* and $\emptyset \in SB$ is an empty soft set and $\widetilde{U}$ be the universal soft set. To show that $(SB, \wedge, \vee, \emptyset, \widetilde{U})$ is a soft Boolean algebra, it is enough to show that the following axioms of Boolean algebra are satisfied:

a. Commutative

Let (F, A), $(G, B) \in (SB, \wedge, \vee, \emptyset, \widetilde{U})$, then

(i) $(F, A) \vee (G, B) = (G, B) \vee (F, A)$

(ii) $(F, A) \wedge (G, B) = (G, B) \wedge (F, A)$

From LHS of (i)

Let $\alpha(x) \in (F, A) \vee (G, B)$

$\Rightarrow x \in A \cup B,$

where $\alpha(x) = F(x) \cup G(x) = G(x) \cup F(x)$

$\Rightarrow x \in B \cup A$

$\Rightarrow \alpha(x) \in (G, B) \vee (F, A).$

Hence,

$$(F, A) \vee (G, B) \widetilde{\subseteq} (G, B) \vee (F, A) \qquad (1)$$

Conversely,

Let $\beta(x) \in (G, B) \vee (F, A)$

$\Rightarrow x \in B \cup A,$

where $\beta(x) = G(x) \cup F(x) = F(x) \cup G(x)$

$\Rightarrow x \in A \cup B$

$\Rightarrow \beta(x) \in (F, A) \vee (G, B).$

Hence,

$$(G, B) \vee (F, A) \widetilde{\subseteq} (F, A) \vee (G, B) \qquad (2)$$

From *(1)* and *(2)*,

$$(F,A) \vee (G,B) = (G,B) \vee (F,A).$$

Similarly, *a* (ii) holds.

b. Associativity

c. Distributive

(i) $(F,A) \vee \big((G,B) \wedge (H,C)\big) = \big((F,A) \vee (G,B)\big) \wedge \big((F,A) \vee (H,C)\big)$
(ii) $(F,A) \wedge \big((G,B) \vee (H,C)\big) = \big((F,A) \wedge (G,B)\big) \vee \big((F,A) \wedge (H,C)\big)$

From LHS of (i)

Let $\alpha(x) \in \Big((F,A) \vee \big((G,B) \wedge (H,C)\big)\Big)$

$\Rightarrow x \in A$ or $\alpha(x) \in \big((G,B) \wedge (H,C)\big)$

i.e $x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow x \in A \cup (B \cap C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$,

where $\alpha(x) = \Big(\big(F(x) \cup G(x)\big) \cap \big(F(x) \cup H(x)\big)\Big)$

$\Rightarrow \alpha(x) \in \big(((F,A) \vee (G,B)) \wedge ((F,A) \vee (H,C)\big).$

Hence,

$$\Big((F,A) \vee \big((G,B) \wedge (H,C)\big)\Big) \tilde{\subseteq} \big(((F,A) \vee (G,B)) \wedge ((F,A) \vee (H,C)\big). \quad (1)$$

Conversely,

Let $\beta(x) \in \Big(\big((F,A) \vee (G,B)\big) \wedge \big((F,A) \vee (H,C)\big)\Big)$

$\Rightarrow \beta(x) \in \big((F,A) \vee (G,B)\big)$ and $\beta(x) \in \big((F,A) \vee (H,C\big)$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

where $\beta(x) = \big((F(x) \cup G(x)\big) \cap \big((F(x) \cup H(x)\big)$

$= F(x) \cup \big((G(x) \cap H(x)\big)$

$\Rightarrow x \in A \cup (B \cap C)$

$\Rightarrow \beta(x) \in (F,A) \vee \big((G,B) \wedge (H,C)\big).$

Hence,

$$\Big(\big((F,A)\vee(G,B)\big)\wedge\big((F,A)\vee(H,C)\big)\Big) \mathrel{\widetilde{\subseteq}} \Big((F,A)\vee\big((G,B)\wedge(H,C)\big)\Big)$$

(2)

From *(1)* and *(2)*,

$$\Big((F,A)\vee\big((G,B)\wedge(H,C)\big)\Big) = \Big(\big((F,A)\vee(G,B)\big)\wedge\big((F,A)\vee(H,C)\big)\Big).$$

Similarly, *c* (ii) holds.

d. Existence of Identity

(i) $(F,A)\vee\widetilde{\emptyset} = (F,A)$

(ii) $(F,A)\wedge\widetilde{U} = (F,A)$

e. Complement

(i) $(F,A)\vee(F,A)^c = \widetilde{U}$

(ii) $(F,A)\wedge(F,A)^c = \widetilde{\emptyset}$

d. and *e.* are trivial and easy to proof.

Hence, $(SB,\ \wedge,\ \vee,\emptyset,\widetilde{U})$ is a soft Boolean algebra.

Some basic Theorems of Soft Boolean algebra

Theorem 4.1

Let $(SB,\ \wedge,\ \vee,\emptyset,\widetilde{U})$ be a soft Boolean algebra. Then for any $(F,A)\in SB$,

a. Domination law

(i) $(F,A)\vee\widetilde{U} = \widetilde{U}$

(ii) $(F,A)\wedge\widetilde{\emptyset} = \widetilde{\emptyset}$

b. Idempotent law

(i) $(F,A)\vee(F,A) = (F,A)$

(ii) $(F,A)\wedge(F,A) = (F,A)$

c. Absorption law

(i) $(F,A) = (F,A)\vee\big((F,A)\wedge(G,B)\big)$

(ii) $(F,A) = (F,A)\wedge\big((F,A)\vee(G,B)\big)$

d. Complement law

(i) $(F,A)\vee(F,A)^c = \widetilde{U}$

(ii) $(F, A) \wedge (F, A)^c = \widetilde{\emptyset}$

e. Uniqueness of $\widetilde{\emptyset}$ *and* $\widetilde{U}$

(i) $(F, A) \vee \widetilde{\emptyset} = (F, A)$

(ii) $(F, A) \wedge \widetilde{U} = (F, A)$

Proof

a. (i)

From LHS of (i),

Let $\alpha(x) \in (F, A) \vee \widetilde{U}$

$\Rightarrow x \in A \cup U,$

where $\alpha(x) = F(x) \cup \widetilde{U} = \widetilde{U}$

$\Rightarrow \beta(x) \in \widetilde{U}$

Hence,

$$(F, A) \vee \widetilde{U} \mathrel{\widetilde{\subseteq}} \widetilde{U} \qquad \textit{(1)}$$

Conversely,

Let $\beta(x) \in \widetilde{U}$

where $\beta(x) = F(x) \cup \widetilde{U}$

$\Rightarrow x \in A \cup U$

$\Rightarrow \beta(x) \in (F, A) \vee \widetilde{U}$

Hence,

$$\widetilde{U} \mathrel{\widetilde{\subseteq}} (F, A) \vee \widetilde{U} \qquad \textit{(2)}$$

From *(1)* and *(2)* we have

$(F, A) \vee \widetilde{U} = \widetilde{U}$

Similarly, *a.* (ii) follows.

b. (i)
From LHS
Let $\alpha(x) \in (F, A) \vee (F, A)$
$\Rightarrow x \in A \cup A = A$
$\Rightarrow x \in A$
$\Rightarrow \alpha(x) \in (F, A).$

Hence,

$$(F,A) \vee (F,A) \tilde{\subseteq} (F,A) \qquad \textit{(1)}$$

Conversely,

Let $\beta(x) \in (F,A)$

$\Rightarrow x \in A$

$\Rightarrow x \in A \cup A$

$\Rightarrow \beta(x) \in (F,A) \vee (F,A)$

Hence,

$$(F,A) \tilde{\subseteq} (F,A) \vee (F,A) \qquad \textit{(2)}$$

From *(1)* and *(2)*,

$$(F,A) \vee (F,A) = (F,A).$$

Similarly, *b.* (ii) follows.

Finally, the remaining results follow similarly.

Theorem 4.2
Let $(SB,\ \wedge,\ \vee, \emptyset, \widetilde{U})$ be a soft Boolean algebra. For any soft sets $(F,A), (G,B)$ and $(H,C) \in SB$, the following hold.

a. (i) $(F,A) \vee (J,O) = (F,A) \vee (P,K)$

(ii) $(F,A)^c \vee (J,O) = (F,A)^c \vee (P,K)$

b. (i) $(F,A) \wedge (J,O) = (F,A) \wedge (P,K)$

(ii) $(F,A)^c \wedge (J,O) = (F,A)^c \wedge (P,K)$

c. (i) $(F,A) \vee (J,O) \neq (F,A) \wedge (P,K)$

(ii) $(F,A)^c \vee (J,O) \neq (F,A)^c \wedge (P,K)$

d. (i) $(F,A) \vee (J,O) \neq (F,A) \wedge (P,K)$

(ii) $(F,A)^c \vee (J,O) \neq (F,A)^c \wedge (P,K)$

Proof

a (i)

Let $(J, O) = (F, A) \vee \big((G, B) \vee (H, C)\big)$ and

$(P, K) = \big((F, A) \vee (G, B)\big) \vee (H, C)$

We show that

$$(F, A) \vee (J, O) = (F, A) \vee (P, K) \quad (*)$$

$$(F, A)^c \vee (J, O) = (F, A)^c \vee (P, K) \quad (**)$$

From LHS of (*)

Let $\alpha(x) \in (F, A) \vee (J, O)$

$\Rightarrow \alpha(x) \in (F, A) \vee \Big((F, A) \vee \big((G, B) \vee (H, C)\big)\Big)$ *By definition*

$\Rightarrow\ x \in A$ or $\alpha(x) \in \Big((F, A) \vee \big((G, B) \vee (H, C)\big)\Big)$

$\Rightarrow\ x \in A$ or $\alpha(x) \in \big((F, A) \vee (G, B)\big)$ or $\alpha(x) \in (H, C)$

$\Rightarrow\ x \in A$ or $x \in (A \cup B)$ or $x \in C$

$\Rightarrow\ x \in A \cup (A \cup B) \cup C$

Where $\alpha(x) = F(x) \cup (F(x) \cup G(x)) \cup H(x)$

$\Rightarrow\ \alpha(x) \in (F, A) \vee \Big(\big((F, A) \vee (G, B)\big) \vee (H, C)\Big)$

$\Rightarrow\ \alpha(x) \in (F, A) \vee (P, K)$ *By definition*

Hence,

$$(F, A) \vee (J, O) \tilde{\subseteq} (F, A) \vee (P, K) \quad \textit{(1)}$$

Also, from RHS of (*)

Let $\beta(x) \in (F, A) \vee (P, K)$

$\Rightarrow\ \beta(x) \in (F, A) \vee \Big(\big((F, A) \vee (G, B)\big) \vee (H, C)\Big)$, *By definition*

where $\beta(x) = F(x) \cup ((F(x) \cup G(x)) \cup H(x))$

$\beta(x) = F(x) \cup (F(x) \cup (G(x) \cup H(x)))$

$\Rightarrow\ x \in A$ or ($x \in A$ or $x \in B \cup C$)

$\Rightarrow\ x \in A \cup \big(A \cup (B \cup C)\big)$

$\Rightarrow\ \beta(x) \in (F, A) \vee \Big((F, A) \vee \big((G, B) \vee (H, C)\big)\Big)$

$\Rightarrow \beta(x) \in (F, A) \vee (J, O)$ *By definition*

Hence,

$$(F,A) \vee (P,K) \widetilde{\subseteq} (F,A) \vee (J,O) \quad (2)$$

From *(1)* and *(2)*,

$$(F,A) \vee (J,O) = (F,A) \vee (P,K).$$

From LHS of (**)

Let $\alpha(x) \in (F,A)^c \vee (J,O)$

$\Rightarrow \alpha(x) \in (F,A)^c \vee \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big)$ *By definition*

$\Rightarrow\ x \in A^c$ or $\alpha(x) \in \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big)$

$\Rightarrow\ x \in A^c$ or $\alpha(x) \in \Big(\big((F,A) \vee (G,B)\big) \vee (H,C)\Big)$

$\Rightarrow\ x \in A^c$ or $x \in (A \cup B)$ or $x \in C$,

where $\alpha(x) = F^c(x) \cup (F(x) \cup G(x)) \cup H(x)$

$\Rightarrow\ \alpha(x) \in (F,A)^c \vee \Big(\big((F,A) \vee (G,B)\big) \vee (H,C)\Big)$

$\Rightarrow\ \alpha(x) \in (F,A)^c \vee (P,K)$ *By definition*

Hence,

$$(F,A)^c \vee (J,O) \widetilde{\subseteq} (F,A)^c \vee (P,K) \quad (1)$$

Also, form RHS of (**)

Let $\beta(x) \in (F,A)^c \vee (P,K)$

$\Rightarrow\ \beta(x) \in (F,A)^c \vee \Big(\big((F,A) \vee (G,B)\big) \vee (H,C)\Big)$, *By definition*

where $\beta(x) = F^c(x) \cup ((F(x) \cup G(x)) \cup H(x))$

$= F^c(x) \cup (F(x) \cup (G(x) \cup H(x)))$

$\Rightarrow\ x \in A^c$ or ($x \in A$ or $x \in B \cup C$)

$\Rightarrow\ x \in A^c \cup (A \cup (B \cup C))$

$\Rightarrow\ \beta(x) \in (F,A)^c \vee \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big)$

$\Rightarrow \beta(x) \in (F,A)^c \vee (J,O)$ *By definition*

Hence,

$$(F,A)^c \vee (P,K) \widetilde{\subseteq} (F,A)^c \vee (J,O) \quad (2)$$

From *(1)* and *(2)*,

$$(F,A)^c \vee (J,O) = (F,A)^c \vee (P,K).$$

Similarly, *b.* (i) and (ii) follows.

c (i)

Let $(J,O) = (F,A) \vee \big((G,B) \vee (H,C)\big)$ and

$$(P,K) = \big((F,A) \vee (G,B)\big) \vee (H,C)$$

We show that

$$(F,A) \vee (J,O) \neq (F,A) \wedge (P,K) \quad (*)$$

$$(F,A)^c \vee (J,O) \neq (F,A)^c \wedge (P,K) \quad (**)$$

From LHS of (*)

Let $\alpha(x) \in (F,A) \vee (J,O)$

$\Rightarrow \alpha(x) \in (F,A) \vee \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big)$ *By definition*

$\Rightarrow x \in A$ or $\alpha(x) \in \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big)$

$\Rightarrow x \in A$ or $\alpha(x) \in \big((F,A) \vee (G,B)\big)$ or $\alpha(x) \in (H,C)$

$\Rightarrow x \in A$ or $x \in (A \cup B)$ or $x \in C$

$\Rightarrow x \in A \cup (A \cup B) \cup C$

Where $\alpha(x) = F(x) \cup \big((F(x) \cup G(x)) \cup H(x)\big)$

$\Rightarrow \alpha(x) \in (F,A) \vee \Big(\big((F,A) \vee (G,B)\big) \vee (H,C)\Big)$

$\Rightarrow \alpha(x) \in (F,A) \vee (P,K) \neq (F,A) \wedge (P,K)$ *By definition*

Hence,

$$(F,A) \vee (J,O) \widetilde{\not\subset} (F,A) \wedge (P,K) \quad (1)$$

Also, from RHS of (*)

Let $\beta(x) \in (F,A) \wedge (P,K)$

$\Rightarrow \beta(x) \in (F,A) \wedge \Big(\big((F,A) \vee (G,B)\big) \vee (H,C)\Big),$ *By definition*

where $\beta(x) = F(x) \cap ((F(x) \cup G(x)) \cup H(x))$
$= F(x) \cap (F(x) \cup (G(x) \cup H(x)))$

$\Rightarrow x \in A$ and ($x \in A$ or $x \in B \cup C$)

$\Rightarrow x \in A \cap \left(A \cup (B \cup C)\right)$

$\Rightarrow \beta(x) \in (F,A) \wedge \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big)$

$\Rightarrow \beta(x) \in (F,A) \wedge (J,O) \neq (F,A) \vee (J,O)$ *By definition*

Hence,

$$(F,A) \wedge (P,K) \tilde{\not\subset} (F,A) \vee (J,O) \qquad (2)$$

From *(1)* and *(2)*,

$$(F,A) \vee (J,O) \neq (F,A) \wedge (P,K).$$

From LHS of (**)

Let $\alpha(x) \in (F,A)^c \vee (J,O)$

$\Rightarrow \alpha(x) \in (F,A)^c \vee \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big)$ *By definition*

$\Rightarrow x \in A^c$ or $\alpha(x) \in \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big)$

$\Rightarrow x \in A^c$ or $\alpha(x) \in \Big(\big((F,A) \vee (G,B)\big) \vee (H,C)\Big)$
$\Rightarrow x \in A^c$ or $x \in (A \cup B)$ or $x \in C$,

where $\alpha(x) = F^c(x) \cup (F(x) \cup G(x)) \cup H(x)$

$\Rightarrow \alpha(x) \in (F,A)^c \vee \Big(\big((F,A) \vee (G,B)\big) \vee (H,C)\Big)$

$\Rightarrow \alpha(x) \in (F,A)^c \vee (P,K) \neq (F,A)^c \wedge (P,K)$ *By definition*

Hence,
$$(F,A)^c \vee (J,O) \tilde{\not\subset} (F,A)^c \wedge (P,K) \qquad (1)$$

Also, form RHS of (**)
Let $\beta(x) \in (F,A)^c \wedge (P,K)$

$\Rightarrow \beta(x) \in (F,A)^c \wedge \Big(\big((F,A) \vee (G,B)\big) \vee (H,C)\Big)$, *By definition* where $\beta(x) = F^c(x) \cap ((F(x) \cup G(x)) \cup H(x))$

$= F^c(x) \cap (F(x) \cup (G(x) \cup H(x)))$

$\Rightarrow x \in A^c$ and ($x \in A$ or $x \in B \cup C$)

$\Rightarrow x \in A^c \cap (A \cup (B \cup C))$

$\Rightarrow \beta(x) \in (F,A)^c \wedge \big((F,A) \vee \big((G,B) \vee (H,C)\big)\big)$

$\Rightarrow \beta(x) \in (F,A)^c \wedge (J,O) \neq (F,A)^c \vee (J,O)$ *By definition*

Hence,

$(F,A)^c \wedge (P,K) \tilde{\not\subset} (F,A)^c \vee (J,O)$ *(2)*

From *(1)* and *(2)*,

$(F,A)^c \wedge (J,O) \neq (F,A)^c \vee (P,K)$.

Similarly, *d.* (i) and (ii) follows.

Theorem 4.3

In the definition of Boolean algebra SB, the associativity laws can be derived from the remaining laws.

Proof

From *Theorem 4.2*

(i) $(F,A) \vee (J,O) = (F,A) \vee (P,K)$

(ii) $(F,A)^c \vee (J,O) = (F,A)^c \vee (P,K)$

We now show that $(J,O) = (P,K)$

i.e. $(F,A) \vee \big((G,B) \vee (H,C)\big) = \big((F,A) \vee (G,B)\big) \vee (H,C)$ (***)

Now, from LHS of (***)

$(F,A) \vee \big((G,B) \vee (H,C)\big) = \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big) \wedge \tilde{U}$ *Identity law*

$= \Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big) \wedge \big((F,A) \vee (F,A)^c\big)$ *Complement law*

$= \Big(\Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big) \wedge (F,A)\Big) \vee \Big(\Big((F,A) \vee \big((G,B) \vee (H,C)\big)\Big) \wedge$
$(F,A)^c\Big)$ *Distributive law*

$= \Big(\big(((F,A) \vee (G,B)) \vee (H,C)\big) \wedge (F,A)\Big) \vee \Big(\big(((F,A) \vee (G,B)) \vee (H,C)\big) \wedge (F,A)^c\Big)$ *Commutative law*

$= \big(((F,A) \vee (G,B)) \vee (H,C)\big) \wedge \big((F,A) \vee (F,A)^c\big)$ *Distributive law*

$= \big(((F,A) \vee (G,B)) \vee (H,C)\big) \wedge \widetilde{U}$ *Identity*

$= \big((F,A) \vee (G,B)\big) \vee (H,C)$

Hence,

$$(F,A) \vee \big((G,B) \vee (H,C)\big) = \big((F,A) \vee (G,B)\big) \vee (H,C)$$

Conclusion

We crisply summarized the basic concepts of fundamentals of soft set. By redefining the concepts of conjunction and disjunction as binary operations on soft sets, many properties were presented. In particular, a perception named soft Boolean algebra is introduced where some related results were established. It is easy to see that this can be extended to many other algebraic structures.

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