NAUB JOURNAL OF SCIENCE AND TECHNOLOGY
(NAUBJOST)
e-ISSN:2811-2350
Email: jost@naub.edu.ng http://journal.naub.edu.ng/naubjournal/

EVALUATION OF SOME COMPLICATED INTEGRALS OF REAL VALUED FUNCTIONS USING CAUCHY'S RESIDUE THEOREM

[1]Dayyabu A., [1]Usman, S., [1]Abdullahi, I. [2]Yusuf, H. B., [1]Agaie, B. G.
[1]Federal University Dutse (FUD), P.M.B 7156 Dutse, Jigawa State, Nigeria
[2]Department of mathematics, Nigerian Army University Biu, P.M.B 1500 Biu, Borno State, Nigeria
Corresponding author: iabdullahi94@gmail.com, ibrahim.abdullahi@fud.edu.ng

Abstract

Complex analysis is considered as one of the powerful tools in solving problems in mathematics, physics, and engineering. In the mathematical field of complex analysis, contour integration is a method of evaluating certain integrals along paths in the complex plane. Cauchy's residue theorem is a powerful tool to evaluate complex integrals of analytic function over closed curves. This paper discussed how various type of definite integrals of real valued function can be associated with integrals around closed curves in the complex plane, so that the residue theorem will become a handy tool for some definite integrals.

Keywords: Complex Analysis, Complex Plane, Real Valued Function, Cauchy's Residue Theorem.

Introduction

Residue or in some texts residuum is a Latin word meaning remaining, that is what is left after a part is taken away.

Cauchy is considered as a principal founder of complex function theory. However, the brilliant Swiss mathematician Euler (1707-1783) also took an important role in the field of complex analysis. The symbol i for $\sqrt{-1}$ was first used by Euler, who also introduced π as the ratio of the length of a circumference of a circle to its diameter and e as a base for the natural logarithms, respectively. He also developed one of the most useful formulas in mathematics. That is Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, which can be derived from De Moivre's theorem.

Complex integration has been advanced by a reason of being evaluated by the method of which many complicated real and complex integrals can be evaluated. The Complex integration make us to evaluate certain real integrals appearing in applications that are not accessible by real integral calculus. The main method of complex integration is Cauchy's integral formula and integration by residue

The residue theorem sometimes called Cauchy's residue theorem is a powerful tool to evaluate real integrals of analytic function over closed curves. Residue theorem has been an important in such a way that its application to the solution of definite integral is a great achievement of complex analysis.

Residue theorem as one of the research field in modern analysis which was established in 19th century, a French mathematician A.L. Cauchy (1789-1857) works on the subject residue with greatly advancing in a way that would help the engineers or scientists. As theorem of residue bearing his name is sufficiently important to merit in the field of complex analysis. Others were German Mathematicians B. Riemann (1826-1866) and K. Weierstrass (1815-1897) continue the work on this field.

Residue theorem has application when integrating along the real line. Some real integrals cannot be evaluated by normal calculus; this is because the integrand does not have a simple anti-derivative. However, we can evaluate them using complex variable and the residue theorem. This is one of the most important applications of the theorem of residue.

Murray *et al* (2009) computed the integral of some rational function of sine and cosine by using Green's theorem.

Ikegami *et al* (2010) collected and sorted out correlated information, He started from Cauchy's integral theorem and Cauchy's integral formula, concluding their complex variable functions internal connections and summarized singular point contour Cauchy's integral formula and find the relationship between residue theorem with Cauchy's integral theorem and Cauchy's integral formula.

Abdulsattar (2017) applied the powerful technique of Contour integral in the complex plane to evaluate some improper integrals. These integrals are very difficult to tackle with the regular calculus techniques of real variables. He used the integration along a branch cut, and the residue theorem, plus the proper choice of contours, to solve interesting integrals.

Methodology

In this work, the concept of residue theorem is applied to evaluate some definite integral of real valued function of real variables. The concept is applied to evaluate real integrals of:

1. Rational function of $\cos\theta$ and $\sin\theta$.
2. Certain types of improper integral,

where basic substitution will be employed to change the given real function to a complex function which can be evaluated using Cauchy's residue theorem.

Theorem 1: Cauchy's Residue theorem

Suppose that $f(z)$ is analytic on and inside a simple closed path C, except at a finite number of points $z_1, z_2, \dots, z_n$ each of which is an isolated singularity of f. Then,

$$\int_C f(z) = 2\pi i \sum_{k=1}^{n} Res\{f, z_k\}$$
$$= 2\pi i[Res\{f, z_1\} + Res\{f, z_2\} + \cdots + Res\{f, z_n\}]$$

For the proof see [John (1997)]

Problem 1: Use residue theorem to evaluate:

$$\int_C \frac{z-2}{z(z-1)} dz, \qquad C: |z| = 2$$

Solution

The function $\frac{z-2}{z(z-1)}$ has two simple poles, one at the origin and the other at $= 1$, both inside C.

$$\therefore \int_C \frac{z-2}{z(z-1)} dz = 2\pi i[Res\{f, 0\} + Res\{f, 1\}]$$

$$and \;\; Res\{f, 0\} = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z(z-2)}{z(z-1)}$$

$$= \lim_{z \to 0} \frac{z-2}{z-1} = \frac{-2}{-1} = 2$$

$$also \;\; Res\{f,1\} = \lim_{z \to 1} (z-1)f(z) = \lim_{z \to 1} \frac{(z-1)(z-2)}{z(z-1)}$$

$$= \lim_{z \to 1} \frac{z-2}{z} = \frac{-1}{1} = -1$$

$$\Rightarrow \int_C \frac{z-2}{z(z-1)} dz = 2\pi i[2-1] = 2\pi i$$

Application of Cauchy's Residue Theorem to Evaluation of Real Integrals

This Section shows how the concept of residue theorem is applied to the evaluation of real integrals of:

1.Rational function of $\cos\theta \; and \; \sin\theta$

2. Certain types of improper integral

Integrals of rational function of $\cos\theta \; and \; \sin\theta$

Integrals of this type have the general form:

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta \qquad for \;\; 0 \le \theta \le 2\pi$$

Where $F$is a rational function of $\cos\theta \;\; and \;\; \sin\theta.$

The basic substitution is employed in the evaluation of such integrals as:

$$z = \cos\theta + i\sin\theta, \quad 0 \le \theta \le 2\pi \tag{1}$$

Note that, as θ varies from $0 \; to \; 2\pi$, z describes the unit circle $C\colon |Z| = 1$ in the positive sense.

From equation 1 we have;

$$\frac{1}{Z} = \frac{1}{\cos\theta + i\sin\theta} = \frac{1}{\cos\theta + i\sin\theta} * \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{cos^2\theta - i\sin\theta\cos\theta + i\sin\theta\cos\theta + sin^2\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{cos^2\theta + sin^2\theta} = \frac{\cos\theta - i\sin\theta}{1} = \cos\theta - i\sin\theta$$

$$\Rightarrow \frac{1}{z} = \cos\theta - i\sin\theta \tag{2}$$

By adding equation 1 and 2 we have;

$$z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

$$\Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{3}$$

By subtracting equation 2 from 1 we have;

$$z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$$

$$= \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$$

$$= 2i\sin\theta$$

$\Rightarrow \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ (4)

From equation 1.

$$z = \cos\theta + i\sin\theta$$

$$\Rightarrow \frac{dz}{d\theta} = -\sin\theta + i\cos\theta$$

$$= i\cos\theta + i^2\sin\theta$$

$$= i(\cos\theta + i\sin\theta)$$

$$= iz$$

$\Rightarrow d\theta = \frac{dz}{iz}$ (5)

Now by substituting equation 3,4 and 5 in to the given integrand yield an integral $\int_c g(z)dz$ which can be evaluated using cauchy's residue theorem.

Theorem 2:

Let F be a rational function such that $F(\cos\theta, \sin\theta)$ is defined for all $\theta \in [0,2\pi]. Then$

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta = 2\pi i \sum_{k=1}^{n} Res\{f, z_k\}$$

Where $f(z) = \frac{1}{iz}F\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)$ $and\ z_1, z_2, \dots, z_n\ are$ the poles of C inside the unit circle $|z| = 1$.

For the proof see [John (1997)]

Problem 2: Evaluate

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$$

Solution

Substitute$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ $and\ d\theta = \frac{dz}{iz}$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{|z|=1} \frac{1}{2+\frac{1}{2}(\frac{z^2+1}{z})} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{1}{2 + \frac{z^2+1}{2z}} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{1}{(4z + z^2 + 1)/2z} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{2z}{(z^2 + 4z + 1)} \frac{dz}{iz}$$

$$= \frac{2}{i} \int_{|z|=1} \frac{1}{z^2 + 4z + 1} dz$$

$$\Rightarrow f(z) = \frac{1}{z^2 + 4z + 1}$$

$z^2 + 4z + 1$ has two simple roots $-2 \pm \sqrt{3}$, of which only $-2 + \sqrt{3}$ lies inside $|z| = 1$, and then, using **Theorem 2** to have;

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2}{i}(2\pi i) Res\{f, -2 + \sqrt{3}\}$$

$$\therefore Res\{f, -2 + \sqrt{3}\} = \lim_{z \to -2+\sqrt{3}} \frac{[z - (-2 + \sqrt{3})]}{z^2 + 4z + 1}$$

$$= \lim_{z \to -2+\sqrt{3}} \frac{[z - (-2 + \sqrt{3})]}{[z - (-2 + \sqrt{3})][z - (-2 - \sqrt{3})]}$$

$$= \lim_{z \to -2+\sqrt{3}} \frac{1}{z + 2 + \sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2}{i}(2\pi i)\left[\frac{1}{2\sqrt{3}}\right]$$

$$= \frac{2\pi}{\sqrt{3}}.$$

Problem 3: Evaluate

$$\int_0^{2\pi} \frac{sin^2\theta}{5 - 4\cos\theta} d\theta$$

Solution:

Substitute $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)\ and\ d\theta = \frac{dz}{iz}$in the given integrand we have:

$$\int_0^{2\pi} \frac{sin^2\theta}{5 - 4cos\theta} d\theta = \int_{|z|=1} \left(\frac{[{(z^2-1)}/{2iz}]^2}{5 - 4[{(z^2+1)}/{2z}]}\right) \frac{dz}{iz}$$

$$= \int_{|z|=1} \left(\frac{(z^2-1)^2/_{-4z^2}}{5-2[(z^2+1)/_z]}\right)\frac{dz}{iz}$$

$$= \int_{|z|=1} \left(\frac{(z^4-2z^2+1)/_{-4z^2}}{(5z-2z^2-2)/_z}\right)\frac{dz}{iz}$$

$$= \int_{|z|=1} \left(\frac{z^4-2z^2+1}{-4z(5z-2z^2-2)}\right)\frac{dz}{iz}$$

$$= -\frac{1}{4i}\int_{|z|=1} \left(\frac{z^4-2z^2+1}{z^2(5z-2z^2-2)}\right)dz$$

$$= \frac{1}{4i}\int_{|z|=1} \left(\frac{z^4-2z^2+1}{z^2(2z^2-5z+2)}\right)dz$$

$$= \frac{1}{4i}\int_{|z|=1} \left(\frac{z^4-2z^2+1}{z^2(z-2)(2z-1)}\right)dz$$

$$\Rightarrow f(z) = \frac{z^4-2z^2+1}{z^2(z-2)(2z-1)}$$

Clearly the integrand $f(z)$has a pole of order 2 at $z = 0$ [in interior of the circle $|z| = 1$].A simple pole at $z = \frac{1}{2}$ [in interior of the circle $|z| = 1$] and a simple pole at $z = 2$ [in exterior of the circle $|z| = 1$]. By using **Theorem 2,**

$$\int_0^{2\pi} \frac{sin^2\theta}{5-4cos\theta}d\theta = \frac{1}{4i}(2\pi i)\left[Res\{f,0\} + Res\left\{f,\frac{1}{2}\right\}\right]$$

We then find the $Res\{f,0\}$ *and* $Res\left\{f,\frac{1}{2}\right\}$.

$$Res\{f,0\} = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d^{2-1}}{dz^{2-1}}\left[\frac{z^2(z^4-2z^2+1)}{z^2(2z^2-5z-2)}\right]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4-2z^2+1}{2z^2-5z-2}\right]$$

$$= \lim_{z\to 0}\left[\frac{(2z^2-5z-2)(4z^3-4z)-(z^4-2z^2+1)(4z-5)}{(2z^2-5z-2)^2}\right]$$

$$= \frac{(-2)(0)-(1)(-5)}{4}$$

$$= \frac{5}{4}.$$

$$And\ also, \qquad Res\left\{f,\frac{1}{2}\right\} = \lim_{z\to\frac{1}{2}}\frac{(z-\frac{1}{2})(z^4-2z^2+1)}{z^2(z-2)(2z-1)}$$

$$= \lim_{z\to\frac{1}{2}} \frac{(2z - {}^{1}\!/_{2})(z^4 - 2z^2 + 1)}{z^2(z-2)(2z-1)}$$

$$= \frac{1}{2}\lim_{z\to\frac{1}{2}} \frac{(2z-1)(z^4 - 2z^2 + 1)}{z^2(z-2)(2z-1)}$$

$$= \frac{1}{2}\lim_{z\to\frac{1}{2}} \frac{(z^4 - 2z^2 + 1)}{z^2(z-2)}$$

$$= \frac{1}{2}\left[\frac{{}^{1}\!/_{16} - {}^{1}\!/_{2} + 1}{{}^{1}\!/_{4}({}^{1}\!/_{2} - 2)}\right]$$

$$= \frac{1}{2}[-{}^{3}\!/_{2}]$$

$$= -{}^{3}\!/_{4}$$

$$\therefore \int_0^{2\pi} \frac{sin^2\theta}{5 - 4\cos\theta} d\theta = \frac{1}{4i}(2\pi i)\left[\frac{5}{4} - \frac{3}{4}\right]$$

$$= \frac{\pi}{2}({}^{2}\!/_{4})$$

$$= {}^{\pi}\!/_{4}$$

Certain Types of Improper Integrals

When integrating along the real line, some real integrals are difficult to solve by normal calculus. This is because the integrand does not have a simple anti derivative. However, we can evaluate them using complex variable and the residue theorem. By considering integrals of the form:

$$I = \int_{-\infty}^{\infty} F(x)dx$$

Where,

(i)$F(x) = {}^{P(x)}\!/_{Q(x)}$.

(ii)$P(x)$ *and* $Q(x)$ Are polynomials.

(iii)$Q(x)$ Has no real zeros.

(iv) The degree of $P(x)$is at least two less than that of $Q(x)$.

Theorem 3: Let $z_1, z_2, \ldots, z_n \in \{z \in C: Im\{z\} > 0\}$, *and let F be analytic on* $\{z \in C: Im\{z\} \geq 0\} -$ $\{z_1,z_2,\ldots,z_n\}$ with *pole at* $z_1, z_2, \ldots, z_n$. *Suppose* further that $\lim_{|z|\to\infty} |zf(z)| < {}^{\varepsilon}\!/_{\pi}$in $\{z: Im\{z\} \geq 0\}$.Then

$$\int_{-\infty}^{\infty} F(x)dx = 2\pi i \sum_{k=1}^{n} Res\{F, z_k\}$$

For the proof see [Chanman (2003)]

Problem 4: Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

Solution

$$\Rightarrow F(x) = \frac{1}{x^2+1}$$

$$\therefore F(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

Clearly $F(z)$ has a pole at $z_1 = i \, and \, z_2 = -i$ but only $z_1 = i$ has imaginary part greater than zero. By **Theorem 3** we have;

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i Res\{F, i\}$$

$$\therefore Res\{F, i\} = \lim_{z \to i} \frac{z-i}{(z-i)(z+i)}$$

$$= \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i \left[\frac{1}{2i}\right] = \pi.$$

Problem 5: Evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}$$

Solution:

$$\Rightarrow F(x) = \frac{x^2}{(x^2+1)(x^2+4)}$$

$$\therefore F(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

The function $F(z)$ has a simple pole at $z = \pm i \, and \, z = \pm 2i. But \, z = i \, and \, z = 2i$ has the imaginary part greater than zero. By **Theorem 3** we have;

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = 2\pi i [Res\{F, i\} + Res\{F, 2i\}].$$

$$\therefore Res\{F, i\} = \lim_{z \to i} \frac{(z-i)z^2}{(z-i)(z+i)(z^2+4)}$$

$$= \lim_{z \to i} \frac{z^2}{(z+i)(z^2+4)}$$

$$= \frac{-1}{2i(3)} = -\frac{1}{6i}$$

$$And\ also, \qquad Res\{F, 2i\} = \lim_{z \to 2i} \frac{(z-2i)z^2}{(z^2+1)(z-2i)(z+2i)}$$

$$= \lim_{z \to 2i} \frac{z^2}{(z^2+1)(z+2i)}$$

$$= \frac{-4}{(-3)(4i)} = \frac{1}{3i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = 2\pi i \left[-\frac{1}{6i} + \frac{1}{3i}\right]$$

$$= 2\pi i \left[\frac{1}{6i}\right] = \frac{\pi}{3}$$

Validation of the Results.

Using other methods i.e. t-substitution and forward real integration on Problems 2 and 3 respectively to validate the earlier results obtained by Cauchy's Residue Theorem. Consider Problem 2, substituting $t = \frac{\theta}{2}\ and\ u = \tan t$ we have;

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_0^{2\pi} \frac{d\theta}{3cos^2\left(\frac{\theta}{2}\right) + sin^2(\frac{\theta}{2})}$$

$$= \int_0^{\pi} \frac{2dt}{3cos^2t + sin^2t}$$

$$= 2(2)\int_0^{\frac{\pi}{2}} \frac{dt}{3cos^2t + sin^2t}$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{dt}{3cos^2t(1 + \frac{1}{3}(\frac{sin^2t}{cos^2t}))}$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{dt}{cos^2t(1 + \frac{1}{3}tan^2t)}$$

$$= \frac{4}{3}\int_0^{\infty} \frac{cos^2tdu}{cos^2t(1 + \frac{1}{3}u^2)}$$

$$= \frac{4}{3}\int_0^{\infty} \frac{du}{1 + \frac{1}{3}u^2}$$

$$= \frac{4}{3}\int_0^{\infty} \frac{du}{1 + (\frac{1}{\sqrt{3}}u)^2}$$

Let $x = \frac{1}{\sqrt{3}}u$ then $du = \sqrt{3}dx$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{4\sqrt{3}}{3} \int_0^{\infty} \frac{dx}{1 + x^2}$$

$$= \frac{4\sqrt{3}}{3} [\tan^{-1} x|_0^{\infty}$$

$$= \frac{4\sqrt{3}}{3} [\tan^{-1}(\infty) - \tan^{-1} 0]$$

$$= \frac{4\sqrt{3}}{3} \left[\frac{\pi}{2}\right] = \frac{2\pi}{\sqrt{3}}.$$

And also, considering Problem 4, we have;

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx = [\tan^{-1} x|_{-\infty}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(-\infty)$$

$$= \tan^{-1}(\infty) + \tan^{-1}(\infty)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Conclusion

Obviously, Problems 2 and 4 showed that residue formulas may be easier to apply than the real integration. Furthermore, Problems 3 and 5 showed that we can also compute some definite integrals which are difficult to solve using real calculus.

Therefore, we have learnt how the ideas of complex analysis make the solution of complicated integrals of real valued functions easier using computation of residues.

Conflict of Interest

The authors declared here that there is no conflict of interest.

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